Geometrical phases on hermitian symmetric spaces

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Abstract

For simple Lie groups, the only homogeneous manifolds G/K, where K is maximal compact subgroup, for which the phase of the scalar product of two coherent state vectors is twice the symplectic area of a geodesic triangle are the hermitian symmetric spaces. An explicit calculation of the multiplicative factor on the complex Grassmann manifold and its noncompact dual is presented. It is shown that the multiplicative factor is identical with the two-cocycle considered by A. Guichardet and D. Wigner for simple Lie groups.

1 Introduction

Six questions referring to the relationship between coherent states and geometry have been presented in [4]. In the same context, reference [6] was devoted to the following question: find a geometric significance of the phase of the scalar product of two coherent states. An explicit answer to this question for the Riemann sphere was given by Perelomov (cf. reference [31]). Earlier, S. Pancharatnam [30, 32] showed that the phase difference between the initial and final state is $< A|A' >= \exp(-i\Omega_{ABC}/2)$, where Ω_{ABC} is the solid angle subtended by the geodesic triangle ABC on the Poincaré sphere. The holonomy of a loop in the projective Hilbert space is twice the symplectic area of any two-dimensional submanifold whose boundary is the given loop (see [1] and Proposition 5.1 in [27]).

A general answer to the question of the geometric significance of the phase of the scalar product of two coherent state vectors using the coherent state embedding and the so called "Cauchy formulas" was given in [6] and [9]. In reference [6] it was proved that for compact hermitian symmetric spaces the phase of the scalar product of two coherent states is twice the symplectic area of a geodesic triangle determined by the corresponding points on the manifold and the origin of the system of coordinates. A similar result was also obtained in another formulation in [16]. In fact, in reference [6] this result was proved on a restricted class of manifolds: the compact, homogeneous, simply connected Hodge manifolds, which are in the same time naturally reductive. But this class of manifolds considered by me in [6] consists in fact only of the Hermitian symmetric spaces [13]. Indeed, any naturally reductive space with an invariant almost Kähler structure is locally

Hermitian symmetric (cf. Corollary 7 in [18]; see also Corollary 9 in the same reference) and simply connectedness implies Hermitian symmetry. On the other side, the results of reference [6] are also true for other manifolds than compact Hermitian symmetric spaces. For example, the results are true for the Heisenberg-Weyl group [31] as well as for the noncompact dual of the complex Grassmann manifold [7].

An explicit formula was presented for the symplectic area of geodesic triangles on the complex Grassmann manifold [6], and also for its noncompact dual ([7], [8]). Lately, I learned that the formula for the symplectic area on the noncompact Grassmann manifold was found out earlier in the paper [19] devoted to the Gromov's norms. The methods of reference [19] were developed in [16]. Also there are other references on two-cocycles on real simple Lie groups, which are related to the symplectic area of geodesic triangles [21, 20, 22].

A. Guichardet and D. Wigner [21] have proved that a simple Lie group has non-trivial continuous 2-cohomology group $H^2(G,\mathbb{R})$ if and only if G/K admits a G-invariant complex structure, where K is a maximal compact subgroup of G. In this context, let us remained some well known facts (cf. e.g. [17]): If $\mathfrak{g}$ is the Lie algebra of the compact and connected Lie group G, then $H^q(\mathfrak{g})$ is isomorphic with the q^{th} cohomology group $H^q(G)$ with real coefficients and the ring $H(\mathfrak{g})$ is isomorphic with the cohomology ring $H(G)$ of G. If $\mathfrak{g}$ is a semi-simple Lie algebra over a field of characteristic 0, then $H^1(\mathfrak{g}) = \{0\}$, $H^2(\mathfrak{g}) = \{0\}$ and $H^3(\mathfrak{g}) \neq \{0\}$. Moreover, for a simply connected Lie group G, not only $H^1(G) = \{0\}$, but also $H^2(G) = \{0\}$.

In the present paper we give an explicit calculation of the multiplicative factor of representations on the complex Grassmann manifold, which, when expressed in Pontrjagin's coordinates, it is shown to be identical with the two-cocycle considered by A. Guichardet and D. Wigner. The notation and technique for manipulating the Grassmann manifold are that from references [3], [5].

The geometric significance of the 2-cocycle (see below eq. (4.1)) as a symplectic area of a geodesic triangle was found by J.-L. Dupont and A. Guichardet [20]. Using the results of [21, 20, 22] and our results in [6, 7], it follows that: *If G is a simple Lie group and K a maximal compact subgroup, then the only coherent state manifolds G/K for which the phase of the scalar product of two coherent state vectors is twice the symplectic area of a geodesic triangle are the hermitian symmetric spaces.* This remark is a completion of our assertions in [6].

In this context, the following question naturally arise: *For which Lie groups G, which admits coherent state representations* (cf. [26], [29]), *the assertion "the phase of the scalar product of two coherent states is twice the symplectic area of geodesic triangles" is still true?*

Let us also remained that G. Lion and M. Vergne [25] have underlined the relationship between the 2-cocycle of the Segal-Shale-Weil projective representation of the symplectic group $G = Sp(B)$ of the vector space (V, B) and the Maslov index. In the same context we mention also the work of B. Magneron [28].

The problem of symplectic area of geodesic triangles on symmetric spaces was considered also by A. Weinstein [33]. See also the paper [12]. Let us mention also the paper [14].

The present paper is laid out as follows: in §2.1 simplest manifolds on which the phase

of the scalar product is twice the symplectic are presented — the sphere $SU(2)/U(1)$, its noncompact dual $SU(1,1)/U(1)$, and the Heisenberg group — while in §2.2 are reviewed our own results referring to the Grassmann manifold. In §3 we present a calculation of the phase which appears when we multiply two representations on the complex Grassmann manifold and its noncompact dual. Our results are compared with those in references [21], [20],[22] in §4. The necessary formulas referring to the complex Grassmann manifold and its noncompact dual are collected in §5. The definitions of coherent states are as in references [10, 11].

2 Previous results

2.1 The phase of the scalar product of coherent states = (2×) symplectic area - basic examples

a) Let us consider the sphere $S^2 = SU(2)/U(1) = \mathbb{CP}^1$. The commutation relations of the generators of the group $SU(2)$ are

$$[J_0, J_\pm] = \pm J_\pm;\ [J_-, J_+] = -2J_0. \tag{2.1}$$

We denote below with the same latter X the generator of the Lie algebra $\mathfrak{g}$ of the Lie group G and the derived representation $d\pi(X)$ of the unitary irreducible representation π of the group G. The action of the generators on the minimal weight vector e_0 ($e_0 = |j, -j>$) is

$$J_+e_0 \neq 0; J_0e_0 = -je_0; J_-e_0 = 0.$$

The coherent state vectors are

$$e_z = e^{zJ_+}e_0,\ z \in \mathbb{C},$$

and the scalar product is

$$(e_{\bar{z}}, e_{\bar{z}'}) = (1 + z\bar{z}')^{2j} = |\ \cdot\ |e^{i\phi},$$

where the phase ϕ is

$$\phi = \frac{j}{2i} \log \frac{(1 + z\bar{z}')}{(1 + z'\bar{z})}\ . \tag{2.2}$$

It can be checked (caution: not an easy exercise!) that

$$\boxed{\begin{array}{c}\text{the phase } \phi \text{ of the scalar product of two coherent states} =\\ (2\times) \text{ symplectic area of the geodesic triangle}\end{array}} \tag{2.3}$$

where the two-form ω on the sphere is

$$\omega = \frac{\sqrt{-1}}{2} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}\ .$$

The formula (2.2) can be find in the book of Perelomov [31] at p. 63. See also Pancharatnam [30].

A similar formula with (2.2),

$$\phi = -\frac{j}{2i}\log\frac{(1-z\bar{z}')}{(1-z'\bar{z})}\ , \tag{2.4}$$

holds for the noncompact manifold $SU(1,1)/U(1)$, with the same significance (2.3). Here instead of (2.1) we have the commutation relations

$$[K_0, K_\pm] = \pm K_\pm; [K_-, K_+] = 2K_0. \tag{2.5}$$

Taking $e_0 = |k, k>$, then

$$K_+e_0 \neq 0, K_0e_0 = ke_0, K_-e_0 = 0, k = 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$$

The coherent states are

$$e_z = e^{zK_+}e_0; (e_{\bar{z}}, e_{\bar{z}'}) = (1-z\bar{z}')^{-2k},$$

and the two-form ω is

$$\omega = \frac{\sqrt{-1}}{2}\frac{dz\wedge d\bar{z}}{(1-|z|^2)^2}\ .$$

b) Now we consider the Heisenberg-Weyl group $\approx \mathbb{C}$. The canonical commutation relations are

$$[a, a^+] = 1; a^+e_0 \neq 0; ae_0 = 0.$$

The Glauber's coherent states are

$$e_z = e^{za^+}e_0,$$

with the scalar product

$$(e_{\bar{z}}, e_{\bar{z}'}) = e^{z\bar{z}'}.$$

Then $\Im(z\bar{z}') = 2\times$ area of the geodesic triangle$(0, z, z+z')$ (cf. eq. (1.1.17) p.10 in [31]).

2.2 Symplectic area of geodesic triangles on the complex Grassmann manifold and its noncompact dual

We remember the formulas which generalize expression (2.2) ((2.4)) of the symplectic area on the sphere in stereographic coordinates ($SU(1,1)/U(1)$) to the complex Grassmann manifold (respectively, its noncompact dual). The notation referring to the complex Grassmann manifold and its noncompact dual is collected in the Appendix (cf. [3], [5]).

Let us denote the compact Grassmann manifold $G_n(\mathbb{C}^{m+n})$ of n-planes in $\mathbb{C}^{n+m}$ by

$$X_c = G_c/K = SU(n+m)/S(U(n)\times U(m))\ , \tag{2.6}$$

and its noncompact dual by

$$X_n = G_n/K = SU(n,m)/S(U(n)\times U(m))\ . \tag{2.7}$$

The following two theorems are extracted from [6, 8, 7].

Theorem 1. *Let $z, z' \in \mathcal{V}_0 \subset G_n(\mathbb{C}^{m+n})$ (resp. its noncompact dual (2.7)) be described by the Pontrjagin's coordinates Z, Z'. Let $\gamma(0, z, z')$ be the geodesic triangle obtained by joining $0, z, z'$. Then the symplectic area of the surface $\sigma(0, z, z')$ of the geodesic triangle $\gamma(0, z, z')$ is given by*

$$\mathfrak{I}(0, Z, Z') = \int_{\sigma(0,z,z')} \omega = \frac{\epsilon}{4i} \log \frac{\det(\mathbb{1} + \epsilon Z Z'^{+})}{\det(\mathbb{1} + \epsilon Z' Z^{+})} \ . \tag{2.8}$$

$\epsilon = 1$ ($\epsilon = -1$) corresponds to the compact (noncompact) manifold X_c (respectively, X_n).

The main ingredients for proving theorem 1 in the compact case are presented in [6]. A similar calculation can be done in the noncompact case (cf. [7]). Here we just remember that the two-form ω in eq. (2.8) is

$$\omega = \frac{i}{2} \mathrm{Tr}[dZ(\mathbb{1}_n + \epsilon Z^+ Z)^{-1} \wedge dZ^+(\mathbb{1}_m + \epsilon Z Z^+)^{-1}]. \tag{2.9}$$

Let us mention that in the context of Gromov's norm of the Kähler class of symmetric domains, A. Domic and D. Toledo [19] have calculated the symplectic area of the geodesic triangle in the case of the noncompact dual of the Grassmann manifold. Their calculation is based on the Stokes's formula

$$\int_{\Delta} \omega = \int_{\gamma(Q,R)} d^{\mathbb{C}} \rho_P, \tag{2.10}$$

Here Δ is a geodesic simplex in the bounded symmetric domain, which has the vertices P, Q, R and ρ_P is the (unique) potential for the Bergmann metric, i.e. a function such that $dd^{\mathbb{C}}\rho_P = \omega$, with the following properties: A) $\rho_P(P) = 0$; B) ρ_P is invariant under the isotropy group of P; C) $d^{\mathbb{C}}\rho_P = 0$ on geodesics through P. We only want to stress that

Remark 1. *The proof in [6] of eq. (2.8) using Stokes's formula is equivalent with the calculation of A. Domic and D. Toledo [19].*

Recall the definition of Perelomov's coherent state vectors:

$$e_{Z,j} = \exp \sum_{\varphi \in \Delta_n^+} (Z_\varphi F_\varphi^+) j, \qquad \underline{e}_{Z,j} = (e_{Z,j}, e_{Z,j})^{-1/2} e_{Z,j}, \tag{2.11}$$

$$e_{B,j} = \exp \sum_{\varphi \in \Delta_n^+} (B_\varphi F_\varphi^+ - \bar{B}_\varphi F_\varphi^-) j, \quad e_{B,j} := \underline{e}_{Z,j}. \tag{2.12}$$

where Δ_n^+ are the positive noncompact roots, $Z := (Z_\varphi) \in \mathbb{C}^d$ (d = complex dimension of M) are the local coordinates in the maximal neighborhood $\mathcal{V}_0 \subset M$, $F_\varphi^+ j \neq 0, F_\varphi^- j = 0$, $\varphi \in \Delta_n^+$, and j is the extreme (here minimal, see eq. (3.14) below) weight vector of the representation. Note that $\mathcal{V}_0 \approx X_n$ for the noncompact case.

Theorem 2. *Let M be a hermitian symmetric manifold. Let us consider on the manifold of coherent states M the Perelomov's coherent vectors (2.11) in a local chart, corresponding to the fundamental representation π. Let us consider the points $Z, Z' \in \mathcal{V}_0 \subset M$ such that $0, Z, Z'$ is a geodesic triangle. Then the phase Φ_M defined by the relation*

$$(\underline{e}_{Z'}, \underline{e}_Z) = |(\underline{e}_{Z'}, \underline{e}_Z)| \exp(i\Phi_M(Z', Z)) \tag{2.13}$$

is given by twice the integral of the symplectic two-form ω_M of M on the surface $\sigma(0, Z, Z')$ of the geodesic triangle $\gamma(0, Z, Z') \subset M$

$$\Phi_M(Z', Z) = 2 \int_{\sigma(0,Z,Z')} \omega_M. \tag{2.14}$$

Also

$$|(\underline{e}_{Z'}, \underline{e}_Z)| = |(\underline{e}_{i(Z')}, \underline{e}_{i(Z)})| = \cos d_C(i(Z'), i(Z)). \tag{2.15}$$

In the last relation

$$d_C([u], [v]) = \arccos \frac{|\langle u, v\rangle|}{\|u\|\|v\|} \ .$$

Note that Theorem 2, which is Theorem 2b) and Theorem 3 in [6], was obtained in [16] under the form of Theorem 2.1.

We are interested to find other manifolds for which eq. (2.3) still holds. But first we present some calculation of the the phase which appears when we multiply two representations on the complex Grassmann manifold.

3 Multiplicative factors on the complex Grassmann manifold and its noncompact dual

Proposition 1. *Let the noncompact (compact) Grassmann manifold X_n (X_c) parameterized in the B and Z parametrizations (A.3a)-(A.3d), where the parameters are related by equation (A.4). Let σ be the section which associates to the element in $Z \in \mathfrak{p}_+$ the element in $G_{n,c}$ given by the equation (A.3c). Then, to a $Z \in \mathfrak{p}_+$ there corresponds a $g \in G$ such that $g \cdot o = Z$. Let $D(B)$ represents the matrix (A.3a) expressed in the form (A.3c). Then*

$$D(B_1)D(B_2) = D(B_3) \times e^{i\Phi}, \tag{3.1}$$

where the multiplicative factor Φ has the expression

$$e^{i\Phi} = \det\left[(\mathbb{1} - \epsilon Z_1 Z_2^+)(\mathbb{1} - \epsilon Z_2 Z_1^+)^{-1}\right]^{-\epsilon/2}. \tag{3.2}$$

Here $Z_3 = Z(B_3)$, where Z_3 is given by eq. (A.9).

Sketch of the Proof.

We give two proofs.

A) First proof is a matrix calculation. We indicate the main steps.

a) The product $D(B_1)D(B_2)$ in the Z_1, Z_2-parametrization of the form (A.3c) is written down as a four-block matrice

$$D(B_1)D(B_2) = \begin{pmatrix} M & N \\ P & Q \end{pmatrix} \tag{3.3}$$

where

$$\begin{aligned} M &= (\mathbb{1} + \epsilon Z_1 Z_1^+)^{-1/2}(\mathbb{1} - \epsilon Z_1 Z_2^+)(\mathbb{1} + \epsilon Z_2 Z_2^+)^{-1/2}, && (3.4a)\\ N &= (\mathbb{1} + \epsilon Z_1 Z_1^+)^{-1/2}(Z_2 + Z_1)(\mathbb{1} + \epsilon Z_2^+ Z_2)^{-1/2}, && (3.4b)\\ P &= -\epsilon(\mathbb{1} + \epsilon Z_1^+ Z_1)^{-1/2}(Z_1 + Z_2)^+(\mathbb{1} + \epsilon Z_2 Z_2^+)^{-1/2}, && (3.4c)\\ Q &= (\mathbb{1} + \epsilon Z_1^+ Z_1)^{-1/2}(\mathbb{1} - \epsilon Z_1^+ Z_2)(\mathbb{1} + \epsilon Z_2^+ Z_2)^{-1/2}. && (3.4d) \end{aligned}$$

Here we present the calculation of M, the calculations of the other matrices being similar:

$$\begin{aligned} M &= (\mathbb{1} + \epsilon Z_1 Z_1^+)^{-1/2}(\mathbb{1} + \epsilon Z_2 Z_2^+)^{-1/2} - \epsilon Z_1(\mathbb{1} + \epsilon Z_1^+ Z_1)^{-1/2}(\mathbb{1} + \epsilon Z_2^+ Z_2)^{-1/2} Z_2^+ \\ &= (\mathbb{1} + \epsilon Z_1 Z_1^+)^{-1/2}(\mathbb{1} + \epsilon Z_2 Z_2^+)^{-1/2} - \epsilon(\mathbb{1} + \epsilon Z_1 Z_1^+)^{-1/2} Z_1 Z_2^+(\mathbb{1} + \epsilon Z_2 Z_2^+)^{-1/2} \\ &= (\mathbb{1} + \epsilon Z_1 Z_1^+)^{-1/2}(\mathbb{1} - \epsilon Z_1 Z_2^+)(\mathbb{1} + \epsilon Z_2 Z_2^+)^{-1/2}. \end{aligned}$$

b) We write down again a Gauss decomposition of the product in eq. (3.3)

$$\begin{pmatrix} M & N \\ P & Q \end{pmatrix} = \begin{pmatrix} \mathbb{1} & Z' \\ \mathbf{0} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \alpha & \mathbf{0} \\ \mathbf{0} & \beta \end{pmatrix} \begin{pmatrix} \mathbb{1} & \mathbf{0} \\ Z & \mathbb{1} \end{pmatrix}, \tag{3.5}$$

where

$$\begin{aligned} M &= \alpha + Z'\beta Z, && (3.6a)\\ N &= Z'\beta, && (3.6b)\\ P &= \beta Z, && (3.6c)\\ Q &= \beta. && (3.6d) \end{aligned}$$

It results $Z' = NQ^{-1}$, and finally, and it is find that $Z' \equiv Z_3$, where Z_3 has the expression given by eq. (A.9). Now we calculate α in the Gauss decomposition (3.5), using eq. (3.6a):

$$\begin{aligned} \alpha = M - NQ^{-1}P &= (\mathbb{1} + \epsilon Z_1 Z_1^+)^{-1/2}(\mathbb{1} - \epsilon Z_1 Z_2^+)(\mathbb{1} + \epsilon Z_2 Z_2^+)^{-1/2} + \\ &\epsilon(\mathbb{1} + \epsilon Z_1 Z_1^+)^{-1/2}(Z_1 + Z_2)(\mathbb{1} + \epsilon Z_2^+ Z_2)^{-1/2}(\mathbb{1} + \epsilon Z_2^+ Z_2)^{1/2}(\mathbb{1} - \epsilon Z_1^+ Z_2)^{-1} \times \\ &(\mathbb{1} + \epsilon Z_1^+ Z_1)^{1/2}(\mathbb{1} + \epsilon Z_1^+ Z_1)^{-1/2}(Z_1 + Z_2)^+(\mathbb{1} + \epsilon Z_2 Z_2^+)^{-1/2}, \end{aligned} \tag{3.7}$$

so we have for α

$$\alpha = (\mathbb{1} + \epsilon Z_1 Z_1^+)^{-1/2}\Lambda(\mathbb{1} + \epsilon Z_2 Z_2^+)^{-1/2}, \tag{3.8}$$

where

$$\Lambda = \mathbb{1} - \epsilon Z_1 Z_2^+ + \epsilon(Z_1 + Z_2)(\mathbb{1} - \epsilon Z_1^+ Z_2)^{-1}(Z_1 + Z_2)^+. \tag{3.9}$$

In order to find a simpler expression for Λ, we substitute in eq. (3.9) for $Z_1 + Z_2$

$$Z_1 + Z_2 = Z_1(\mathbb{1} - \epsilon Z_1^+ Z_2) + (\mathbb{1} + \epsilon Z_1 Z_1^+)Z_2, \tag{3.10a}$$

and, similarly for its adjoint

$$(Z_1 + Z_2)^+ = (\mathbb{1} - \epsilon Z_1^+ Z_2) Z_2^+ + Z_1^+ (\mathbb{1} + \epsilon Z_2 Z_2^+). \tag{3.10b}$$

For Λ in eq. (3.8) we finally find

$$\Lambda = (\mathbb{1} + \epsilon Z_1 Z_1^+)(\mathbb{1} - \epsilon Z_2 Z_1^+)^{-1}(\mathbb{1} + \epsilon Z_2 Z_2^+). \tag{3.11}$$

c) In accord with the Gauss decomposition (A.3b) of equation (A.3c), we have

$$D(B_3) = \begin{pmatrix} \mathbb{1} & Z_3 \\ \mathbf{0} & \mathbb{1} \end{pmatrix} \begin{pmatrix} U & \mathbf{0} \\ \mathbf{0} & V \end{pmatrix} \begin{pmatrix} \mathbb{1} & \mathbf{0} \\ -\epsilon Z_3^+ & \mathbb{1} \end{pmatrix},$$

$$U = (\mathbb{1} + \epsilon Z_3 Z_3^+)^{1/2}, \tag{3.12}$$

where Z_3 has the expression given by eq. (A.9). The calculation is similar with equations (8.6) and (8.7) in [5], where in eq. (8.3) we have to substitute $Z_1 \rightarrow -Z_1$. This corresponds to the fact: $U^{-1}(Z) = U(-Z)$ (see the Appendix).

The final expression needed to determine U in eq. (3.12) is:

$$\begin{aligned} \mathbb{1} + \epsilon Z_3 Z_3^+ \;\; = \;\; & (\mathbb{1} + \epsilon Z_1 Z_1^+)^{1/2} (\mathbb{1} - \epsilon Z_2 Z_1^+)^{-1} \times \\ & (\mathbb{1} + \epsilon Z_2 Z_2^+)(\mathbb{1} - \epsilon Z_1 Z_2^+)^{-1} (\mathbb{1} + \epsilon Z_1 Z_1^+)^{1/2}. \end{aligned} \tag{3.13}$$

d) We use the following relation (see the case of the maximal weight for the compact case e.g. eq. (3.12) in [2]) for the action of the representation π on the minimal weight vector o:

$$j_1 = j_2 = \cdots j_n = k, j_{n+1} = j_{n+2} = \cdots j_{n+m} = 0, \tag{3.14}$$

$$\pi \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} o = (\det A)^{-k\epsilon} \; o, \;\; \det A \det D = 1. \tag{3.15}$$

The sphere $SU(2)/U(1)$ corresponds to $k = 2j, j = \frac{1}{2}, 1, \cdots$, while the dual noncompact case $SU(1,1)/U(1)$ is obtained replacing $k \rightarrow 2k = 2, 3, \cdots$. For the case of the complex Grassmann manifold and its noncompact dual below we take $k = 1$ if not specified otherwise. The phase Φ in eq. (3.1) is obtained from the relation

$$(\det \alpha)^{-\epsilon} = (\det U)^{-\epsilon} e^{i\Phi},$$

where α is given by the equations (3.8), (3.11) and U by equations (3.12), (3.13). ■

B). We now present briefly a second proof of Proposition 1. We recall firstly some general considerations on multipliers and coherent states [10], [11]. Here again π is unitary irreducible representation of the group G on a Hilbert space $\mathcal{H}$.

Let

$$f_\psi(z) = (e_{\bar{z}}, \psi) = \frac{(\pi(\bar{g})e_0, \psi)}{(\pi(\bar{g})e_0, e_0)}, \;\; z \in M, \psi \in \mathcal{H}. \tag{3.16}$$

We get

$$f_{\pi(\overline{g'}).\psi}(z) = \mu(g', z) f_\psi(\overline{g'}^{-1}.z), \tag{3.17}$$

where

$$\mu(g',z)=\frac{(\pi(\overline{g'}^{-1}\overline{g})e_0,e_0)}{(\pi(\overline{g})e_0,e_0)}=\frac{\Lambda(g'^{-1}g)}{\Lambda(g)}. \tag{3.18}$$

We recall that

$$\pi(g).e_0=e^{i\alpha(g)}e_{\tilde{g}}=\Lambda(g)e_{z_g} \tag{3.19}$$

where we have used the decompositions, $g=\tilde{g}.h$, $(G=G/H.H)$; $g=z_g.b$ $(G_{\mathbb{C}}=G_{\mathbb{C}}/B.B)$. We have also the relation $\chi_0(h)=e^{i\alpha(h)}$, $h\in H$ and $\chi(b)=\Lambda(b)$, $b\in B$, where $\Lambda(g)=\frac{e^{i\alpha(g)}}{(e_{\bar{z}},e_{\bar{z}})^{1/2}}$. We can also write down another expression for the multiplicative factor μ appearing in eq. (3.17) using CS-vectors

$$\mu(g',z)=\Lambda(\bar{g'})(e_{\bar{z}},e_{\bar{z'}})=e^{i\alpha(\bar{g'})}\frac{(e_{\bar{z}},e_{\bar{z'}})}{(e_{\bar{z'}},e_{\bar{z'}})^{1/2}} \tag{3.20}$$

and

$$\arg\mu(g',z)=\alpha(\bar{g'})+\Phi_M(\bar{z},\bar{z'}).$$

The following assertion is easy to be checked using successively eq. (3.18):

Remark 2. *Let us consider the relation (3.16). Then we have (3.17), where μ can be written down as in equations (3.18), (3.20). We have the relation*

$$\mu(g,z)=J(g^{-1},z)^{-1}, \tag{3.21}$$

i.e. the multiplier μ is the cocycle in the unitary representation $(\pi_K,\mathcal{H}_K)$ attached to the positive definite holomorphic kernel $K(z,\bar{w}):=(e_{\bar{z}},e_{\bar{w}})$

$$(\pi_K(g).f)(x):=J(g^{-1},x)^{-1}.f(g^{-1}.x), \tag{3.22}$$

and the cocycle verifies the relation

$$J(g_1g_2,z)=J(g_1,g_2z)J(g_2,z). \tag{3.23}$$

Note that *the prescription (3.22) defines a continuous action of G on* $\mathrm{Hol}(M,\mathbb{C})$ *with respect to the compact open topology on the space* $\mathrm{Hol}(M,\mathbb{C})$. *If $K:M\times\bar{M}\to\mathbb{C}$ is a continuous positive definite kernel holomorphic in the first argument satisfying*

$$K(g.x,\overline{g.y})=J(g,x)K(x,\bar{y})J(g,y)^*, \tag{3.24}$$

$g\in G$, $x,y\in M$, then the action of G leaves the reproducing kernel Hilbert space $\mathcal{H}_K\subseteq$ $\mathrm{Hol}(M,\mathbb{C})$ *invariant and defines a continuous unitary representation $(\pi_K,\mathcal{H}_K)$ on this space* (cf. Prop. IV.1.9 p. 104 in Ref. [29]).

In Perelomov's notation at p.42 in [31], eqs. (3.17)-(3.20) which define the multiplicative factor μ read

$$\pi(g_1)e_B=e^{i\beta(g_1,z)}e_{g_1.B}. \tag{3.25}$$

Let us recall the notation $(e_z,e_{z'})=e^{i\Phi(z,z')}|(e_z,e_{z'})|$. With the Remark 2, it is easy to see that

$$\beta(g_1,z)=\Phi(z_{g_1^{-1}},z_g)-\alpha(g_1^{-1}). \tag{3.26}$$

It can be proved that the phase α on $X_{c,n} = G_{c,n}/S(U(n)\times U(m)$ is given by

$$\pi \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} e_0 = \left[\frac{\det(A_1)}{\overline{\det}(A_1)}\right]^{-k\frac{\epsilon}{2}} e_B, \quad \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in G_{c,n}. \tag{3.27}$$

$B \in \mathfrak{g}_{c,n}$ in eq. 3.27 is given by eq. (A.5) with $Z = B_1 D_1^{-1}$ or

$$ZZ^+ = \epsilon[(A_1A_1^+)^{-1} - 1].$$

But if the matrix of the group G_c (G_n) is taken from the homogeneous space X_c (respectively, X_n), then in eq. (3.26) $\alpha = 0$. Then it is used the relation $(e_z, e_{z'}) = \det(\mathbb{1} + z'z^+)^\epsilon$ and eq. (3.2) is re-obtained.

It can be seen that eq. (3.2) can be deduced from the equation:

$$\pi \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} e_B = \left[\frac{\det(A_1 - \epsilon B_1 Z^+)}{\overline{\det}(A_1 - \epsilon B_1 Z^+)}\right]^{-k\frac{\epsilon}{2}} e_{g_1.B}. \tag{3.28}$$

We take $k = 1$ and the matrix g_1 is taken of the form given by eq. (A.3c). ■

So, in this section we have presented a brute-force calculation of the multiplicative factor (3.1) and a simpler proof of the same calculation. In the next section the exact meaning of this cocycle will be clarified.

We end the section with another

Remark 3. *The relation (3.24) can be used to find the cocycle J in equation (3.22).*

We illustrate this assertion by the example of the complex Grassmann manifold $G_n(\mathbb{C}^{m+n})$ and its noncompact dual. The scalar product (the reproducing kernel) corresponding to the extremal weight (3.14) is

$$K(X,\bar{Y}) = (e_{\bar{X}}, e_{\bar{Y}}) = \det(\mathbb{1} + \epsilon XY^+)^{\epsilon k}. \tag{3.29}$$

Below we take $k = 1$. Then

$$K(g.X, g.Y) = \det(XB^+ - \epsilon A^+)^{-\epsilon} K(X,Y) \overline{\det}(YB^+ - \epsilon A^+)^{-\epsilon}.$$

$$J(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, X) = \det(A^+ - \epsilon XB^+)^{-\epsilon}$$

is an automorphy factor (see e.g. [15]) and eq. (3.23) is satisfied. If

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(n+m) \text{ or } SU(n,m),$$

then

$$g^{-1} = \begin{pmatrix} A^+ & \epsilon C^+ \\ \epsilon B^+ & D^+ \end{pmatrix},$$

and

$$J(\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}, X) = \det(A - XC)^{-\epsilon}. \tag{3.30}$$

Equation (3.22) reads in this case

$$\pi \begin{pmatrix} A & B \\ C & D \end{pmatrix} f(X) = \det(A - XC)^\epsilon f[(A^+X + \epsilon C^+)(\epsilon B^+ X + D^+)^{-1}]. \ \blacksquare$$

4 Two-cocycles and symplectic areas of geodesic triangles on hermitian symmetric spaces

Firstly we review some results obtained by A. Guichardet and D. Wigner, and J.-L. Dupont and A. Guichardet. Then, using their results, we answer partially to the question addressed at the end of §2.2.

a) A. Guichardet and D. Wigner [21] have proved that: *a simple Lie group has non-trivial continuous 2-cohomology group $H^2(G,\mathbb{R})$ if and only if G/K admits a G-invariant complex structure*, where K is a maximal compact subgroup of G. More exactly, the starting point of their investigation is the following lemma (presented here in abbreviated form), based mostly on the results collected in Helgason's book [24]:

Lemma 1. *Let G be simple. Then:*
(a) $[\mathfrak{p},\mathfrak{p}]=\mathfrak{k}$;
(b) the adjoint representation of $\mathfrak{k}$ in $\mathfrak{p}$ is irreducible;
(c) the next conditions are equivalent:
(i) $H^2(G,\mathbb{R})\neq 0$;
(ii) $\mathrm{Hom}_{\mathfrak{k}}(\bigwedge^2\mathfrak{p},\mathbb{R})\neq 0$;
(iii) there is a $\mathfrak{k}$-invariant complex structure;
(iv) G/K admits a G-invariant complex structure;
(v) the center $\mathfrak{Z}(\mathfrak{k})$ of $\mathfrak{k}$ is non void;
(vi) $\mathrm{Hom}(\mathfrak{k},\mathbb{R})\neq 0$;
(d) If the previous conditions are fulfilled, then:
(i') $\dim H^2(G,\mathbb{R})=\dim\mathfrak{Z}(\mathfrak{k})=\dim\mathrm{Hom}(\mathfrak{k},\mathbb{R})=1$;
(ii') the G-invariant complex structure on G/K is hermitian.

Above $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ is a Cartan decomposition.

Here is the well known list of groups G which have real hermitian Lie algebras: $SU(m,n)$; $SO_0(2,q)$; ($q=1$ or $q\geq 3$); $Sp(n,\mathbb{R})$ $(n\geq 1)$; $SO^*(2n)$ $(n\geq 2)$; E_6; E_7.

A. Guichardet and D. Wigner have considered the real differentiable 2-cocycle f:

$$f(g_1,g_2)=\frac{1}{2\pi}\arg(v(g_1)v(g_2)v(g_1g_2)^{-1}), \tag{4.1}$$

where v is a non-trivial morphism of G in the torus T.

Note that $H^s(G,\mathbb{R})=H^s(\mathfrak{g},\mathfrak{k},\mathbb{R})=H^s(\hat{G}/K,\mathbb{R})=\mathrm{Hom}_K(\bigwedge^s\mathfrak{p},\mathbb{R})$, where $\hat{G}$ is the compact form of the real noncompact simple Lie group G (cf. [22]). More exactly, cf. Proposition 7.6 in [22]: *$f(g_1,g_2)$ defines a two-cocycle $f\in Z^2_{diff}(G,\mathbb{R})_K$ whose class in $H^2(G,\mathbb{C})$ corresponds to the element in $H^2(\mathfrak{g},\mathfrak{k},\mathbb{C})$ via the van Est isomorphism.*

Remark 4. *The multiplicative factor Φ in eq. (3.2) is the two-cocycle (4.1) determined by A. Guichardet and D. Wigner for the group $G=G_n=SU(n,m)$ expressed in Pontrjagin's coordinates on X_n $(\epsilon=-1)$.*

Proof. For $g\in G_n$ put in the block form

$$g=\begin{pmatrix} a & b\\ c & d\end{pmatrix} \tag{4.2}$$

it has been shown in [21] that v in eq. (4.1) is given by $v(g) = \det a$. In Pontrjagin's coordinates $Z = \pi(g)$, ($\pi\sigma = 1$, where π is the natural projection $G_{c,n} \to X_{c,n}$), the function v in the cocycle (4.1) is (below, in the formulas from [21] $\epsilon = -1$):

$$v(Z) = v(\pi(g)) = \det(\mathbb{1} + \epsilon Z Z^+)^{-1/2}. \tag{4.3}$$

Multiplying two matrices $g_i = g(Z_i)$, $i = 1, 2$ of the type (4.2) we get a matrix of the same form, where the block of the type a is the M given by eq. (3.4a), i.e.

$$a = (\mathbb{1} + \epsilon Z_1 Z_1^+)^{-1/2}(\mathbb{1} - \epsilon Z_1 Z_2^+)(\mathbb{1} + \epsilon Z_2 Z_2^+)^{-1/2}. \tag{4.4}$$

Combining eqs. (4.3) and (4.4), we get for the 2-cocycle (4.1) the expression

$$f(Z_1, Z_2) = \frac{1}{2\pi} \arg\det(\mathbb{1} - \epsilon Z_1 Z_2^+)^{-1}, \tag{4.5}$$

$$f(Z_1, Z_2) = \frac{1}{4\pi i} \log \frac{\det(\mathbb{1} - \epsilon Z_2 Z_1^+)}{\det(\mathbb{1} - \epsilon Z_1 Z_2^+)}. \quad \blacksquare \tag{4.6}$$

So, Proposition 1 gives the expression (4.6), independent of the results of A. Guichardet and D. Wigner, in Pontrjagin's coordinates. More exactly,

$$e^{i\epsilon\Phi} = e^{-2\pi i f(g_1, g_2)}. \tag{4.7}$$

b) The geometric significance of the 2-cocycle (4.1) was found by J.-L. Dupont and A. Guichardet [20]. Let in their notation v_{*e} be the differential of the homomorphism v at the origin $e \in G$ and $P = v_{*e}/2\pi i$, and let $P(\Omega)$ be the $G-$invariant differential 2-form on G/K with the value of the origin o given by $P(\Omega)_0(A, B) = -\frac{1}{2}P([A, B])$, $A, B \in \mathfrak{p}_n$. Let $\Delta(g_1, g_2)$ be the geodesic cone with corner o and base the geodesic joining $g_1 \cdot o$ with $g_1 g_2 \cdot o$. Then, in a previous publication (see references in [20]) J.-L. Dupont has constructed the 2-cocycles by integration of $G-$invariant differential forms on geodesic simplexes in a symmetric space G/K, where K is a maximal compact subgroup of G

$$c(g_1, g_2) = \int_{\Delta(g_1, g_2)} P(\Omega). \tag{4.8}$$

In the quoted paper [20] J.-L. Dupont and A. Guichardet have proved the equality

Theorem 3.

$$f(g_1, g_2) = c(g_1, g_2), \;\; g_1, g_2 \in G. \tag{4.9}$$

Remark 5. *Now we only express Theorem 3 in Pontrjagin's coordinates.*

In order to calculate $c(g_1, g_2)$, we have to calculate $\mathfrak{I}(0, Z_1, Z_3)$ with formula (2.8), i.e.

$$c(g_1, g_2) = \frac{\epsilon}{4i} \log A(A^+)^{-1}, \tag{4.10}$$

where

$$A = \mathbb{1} + \epsilon Z_1 Z_3^+,$$

and Z_3 is given by eq. (A.9). It is obtained

$$A = (\mathbb{1} + \epsilon Z_1 Z_1^+)^{1/2} B (\mathbb{1} + \epsilon Z_1 Z_1^+)^{-1/2},$$

while for B it is obtained

$$B = (\mathbb{1} - \epsilon Z_1 Z_2^+)^{-1} (\mathbb{1} + \epsilon Z_1 Z_1^+).$$

We get

$$c(g_1, g_2) = \frac{\epsilon}{4i} \log\det \frac{(\mathbb{1} - \epsilon Z_2 Z_1^+)}{(\mathbb{1} - \epsilon Z_1 Z_2^+)}. \tag{4.11}$$

We checked the validity of Theorem 3 and get $f(g_1, g_2) = \frac{\epsilon}{\pi} c(g_1, g_2)$, the multiplicative factor coming from a different normalization. ■

In Theorem 1 we have calculated the expression of the area of the geodesic triangle with vertices $(0, z, z')$, while eq. (4.6) gives the expression of the 2-cocycle f.

Putting together the results of the papers of A. Guichardet and D. Wigner, and J.-L. Dupont and A. Guichardet [21, 20] (see also the book of A. Guichardet [22]), we get an answer to our question of generalizing the relation (2.3) to simple Lie groups:

Theorem 4. *Let G be a simple Lie group and K a maximal compact subgroup. Then the only coherent state manifolds G/K for which the phase of the scalar product of two coherent state vectors is twice the symplectic area of a geodesic triangle are the hermitian symmetric spaces.*

So far, we have seen that between the **simple Lie groups** G, **only** the groups which have a real hermitian simple Lie algebra lead to coherent states based on G/K which have the property (2.3). Meanwhile, the same property (2.3) is verified by the Heisenberg group, as was underlined in §2.1. Hence, it is natural to formulate the following **question: For which groups G which admits coherent state representations the property (2.3) is still true?** We recall that the group G admits coherent state representations (cf. W. Lisiecki [26] and K. Neeb [29]) if $G/H,\ H \subset K$ admits a holomorphic embedding in a projective Hilbert space, where H is isotropy subgroup of the representation with extreme weight vector e_0. For example, property (2.3) is still true if G is a semi-direct product of a hermitian type group (i.e. G/K is hermitian symmetric) and a Heisenberg group? The answer to this question is given by those CS-groups G which lead to naturally reductive homogeneous spaces G/H.

In this context of Theorem 4, we would like to recall our result established in [10, 11]:

Remark 6. *The coherent state manifold $M \cong G/H$, for which the isotropy representation has discrete kernel, or for admissible Lie algebras and faithful CS-representations, is a reductive space.*

In the same context, let us recall the following classical result [23]:

Theorem 5. *Let G/B be a Kählerian homogeneous space of a reductive Lie group G and let G be effective on G/B. If the Riemannian connection on G/B induced by the invariant Kählerian metric is the canonical affine connection of the first kind with respect to a certain B-invariant decomposition of $\mathfrak{g}$, then G/B is hermitian symmetric.*

5 Appendix: parametrization of the Grassmann manifold and its noncompact dual

The elements $U \in G_{n,c}$ verify the relation

$$U^+ I_{nm}(\epsilon) U = I_{nm}(\epsilon), \quad I_{nm}(\epsilon) = \begin{pmatrix} \epsilon \mathbb{1}_n & \mathbf{0} \\ \mathbf{O} & \mathbb{1}_m \end{pmatrix}, \tag{A.1}$$

where $\epsilon = 1(-1)$ corresponds to G_c (resp. G_n).

Also we have the Cartan decomposition

$$\mathfrak{g}_{n,c} = \mathfrak{k} + \mathfrak{p}_{n,c}; \tag{A.2}$$

$\mathfrak{g}_{n,c}$ ($\mathfrak{k}$) denotes the Lie algebra of the group $G_{n,c}$ (respectively K), and we have, locally (globally), the diffeomorphism of X_c with $\mathfrak{p}_c$ (respectively, X_n) with $\mathfrak{p}_n$

$$X_{n,c} = \exp(\mathfrak{p}_{n,c}) o.$$

The manifold X_c and its noncompact dual X_n is parametrized by $B \in \mathfrak{p}_{n,c}$

$$\begin{aligned}
X_{n,c} &= \exp \begin{pmatrix} \mathbf{0} & B \\ -\epsilon B^+ & \mathbf{0} \end{pmatrix} o = \begin{pmatrix} \mathrm{co}\sqrt{BB^+} & B\dfrac{\mathrm{si}\sqrt{B^+B}}{\sqrt{B^+B}} \\ -\epsilon\dfrac{\mathrm{si}\sqrt{B^+B}}{\sqrt{B^+B}} B^+ & \mathrm{co}\sqrt{B^+B} \end{pmatrix} o & \text{(A.3a)} \\
&= \begin{pmatrix} \mathbb{1} & Z \\ \mathbf{0} & \mathbb{1} \end{pmatrix} \begin{pmatrix} (\mathbb{1} + \epsilon ZZ^+)^{1/2} & \mathbf{0} \\ \mathbf{0} & (\mathbb{1} + \epsilon Z^+ Z)^{-1/2} \end{pmatrix} \begin{pmatrix} \mathbb{1} & \mathbf{0} \\ -\epsilon Z^+ & \mathbb{1} \end{pmatrix} o & \text{(A.3b)} \\
&= \begin{pmatrix} (\mathbb{1} + \epsilon ZZ^+)^{-1/2} & Z(\mathbb{1} + \epsilon Z^+ Z)^{-1/2} \\ -\epsilon(\mathbb{1} + \epsilon Z^+ Z)^{-1/2} Z^+ & (\mathbb{1} + \epsilon Z^+ Z)^{-1/2} \end{pmatrix} o, & \text{(A.3c)} \\
&= \exp \begin{pmatrix} \mathbf{0} & Z \\ \mathbf{0} & \mathbf{0} \end{pmatrix} P, & \text{(A.3d)}
\end{aligned}$$

where $\epsilon = 1(-1)$ for compact (respectively non-compact) manifolds. Here co is the circular cosine cos (resp. the hyperbolic cosine coh) for X_c (resp. X_n) and similarly for si. Z is the $n \times m$ matrix of Pontrjagin's coordinates in $\mathcal{V}_0$ related to B by the formula

$$Z = Z(B) = B\frac{\mathrm{ta}\sqrt{B^+B}}{\sqrt{B^+B}}, \tag{A.4}$$

and ta - the hyperbolic tangent tanh (resp. the circular tangent tan) for X_n (resp. X_c). The relation inverse to eq. (A.4) is

$$B = \frac{\mathrm{arcta}\sqrt{ZZ^+}}{\sqrt{ZZ^+}} \tag{A.5}$$

The transitive action of an element of the group $G_c = SU(n+m)$ ($G_n = SU(n,m)$) on X_c (resp. X_n) is given by the linear fractional transformation

$$Z' = Z'(Z) = U \cdot Z = (AZ+B)(CZ+D)^{-1},\ U = \left(\begin{array}{cc} A & B \\ C & D \end{array} \right) \in G_c(G_n). \tag{A.6}$$

So, we have to find a matrix $U \in G_c(G_n)$ such that eq. (A.1) is satisfied, i.e.

$$\left\{ \begin{array}{rcl} A^+A + \epsilon C^+C & = & \mathbb{1}_n, \\ \epsilon B^+B + D^+D & = & \mathbb{1}_m, \\ \epsilon B^+A + D^+C & = & \mathbf{0}. \end{array} \right. \tag{A.7}$$

Now let $g_i \cdot 0 = Z_i$, $i = 1, 2$, and let

$$Z_3 := g_1 \cdot Z_2 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} (Z_1) \cdot Z_2 = (A_1Z_2 + B_1)(C_1Z_2 + D_1)^{-1}. \tag{A.8}$$

The matrix appearing in eq. (A.3c), a solution of eq. (A.7), fixes a section $\sigma : G/K \rightarrow G$ such that $\sigma(o) = e$. It can be shown (similar to eq. (8.3) in [5], but with $Z_1 \rightarrow -Z_1$) that Z_3 has the expression (A.9):

$$Z_3 = (\mathbb{1} + \epsilon Z_1Z_1^+)^{-1/2}(Z_1 + Z_2)(\mathbb{1} - \epsilon Z_1^+Z_2)^{-1}(\mathbb{1} + \epsilon Z_1^+Z_1)^{1/2}. \tag{A.9}$$

We mention also the following useful relation (which enables to get eq. (A.9) using eq. (8.3) in [5]): *Let $\sigma : X_{n,c} \rightarrow G_{n,c}$ be the section with the property that $\sigma(o) = e$ which associates to a point in the Grassmann manifold $X_{n,c}$ the matrix U (A.6) given by (A.3c) such that $U \cdot o = Z \in X_{n,c}$. Then $U^{-1}(Z) = U(-Z)$.*

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